The aforementioned experimental results suggest that $Nu_{L,\max}$ might be related with stall length. Figure 2 shows the plotted result of $Nu_{L,\max}/Re_L^{2/3}$ vs x_R/L. It may be concluded that the data are well correlated by the following expression.

$$Nu_{L,\max} = [0.446 - 0.238(x_R/L)^{0.161}]\, Re_L^{2/3}. \qquad (4)$$

In Fig. 2, the experimental data by Krall *et al.* and Filetti *et al.* are compared with authors' results. The slopes of Filetti's results show a good agreement with authors' in the limited range of x_R/L. It is interesting to note that $Nu_{L,\max}$ is approximated by (4) without any distinction of short or long stall. In conclusion, $Nu_{L,\max}$ is solely dependent on a stall length, x_R/L, and decreases with increasing stall length. The accuracy of $Nu_{L,\max}$ estimated by the empirical relation (4) is within $\pm 10\%$ for $4 \times 10^3 < Re_L < 8 \times 10^4$ and $0.2 < x_R/L < 16.0$.

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AN INTEGRAL EQUATION APPROACH TO AC DIFFUSION

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NOMENCLATURE

T,	unknown function;
D,	diffusion constant;
i,	imaginary unit;
ω,	angular frequency;
G,	Green's function;
r,	distance in the two dimensional plane;
ker_0, kei_0	Kelvin functions of order zero;
$\bar{u}_n$,	unity vector normal to the boundary;
ρ,	unknown boundary function.

1. INTRODUCTION

WHEREAS the integral equation technique has been widely used for potential and electromagnetic scattering problems [1–8], this technique is not commonly known for other applications. The basic idea for using an integral equation consists in the numerical solution of the problem. A two dimensional partial differential equation will be replaced by a one dimensional integral equation. This fact saves memory storage and computation time. The programming of the problem is then also simplified.

The transient analysis of a thermal diffusion problem by an integral equation has been performed by Shaw [9]. Similar methods have been applied for a drift-diffusion problem [10, 11]. In this paper, an integral equation will be derived for the equation:

$$D\nabla^2 T = i\omega T \qquad (1)$$

which describes the diffusion phenomenon in a two dimensional area S under AC conditions ($\partial/\partial t \to i\omega$).

2. INTEGRAL EQUATION

In order to establish an integral equation for the equation (1), one has to know the Green's function G of the problem. This function is a solution of:

$$\nabla^2 G - \frac{i\omega}{D} G = \delta(\bar{r}) \qquad (2)$$

in the infinite two dimensional plane. One can then use polar coordinates (r, θ) and by taking the circular symmetry into account, the θ-dependence may be dropped. The Green's function G depends only upon the distance r and is found to be:

$$G(r) = \frac{1}{2\pi}\{ker_0[r\sqrt{(\omega/D)}] + i\,kei_0[r\sqrt{(\omega/D)}]\} \qquad (3)$$

where ker_0 and kei_0 are the Kelvin functions of zeroth order [12].

The integral equation technique will now be outlined for the particular geometry presented on Fig. 1. The same method can be applied for arbitrary geometries. The boundary conditions are (Fig. 1):

$$\begin{aligned} T &= T_0 && \text{on } AA' \\ T &= 0 && \text{on } BB' \\ \nabla T \cdot \bar{u}_n &= 0 && \text{on } AB \text{ and } A'B'. \end{aligned} \qquad (4)$$

By using the y-independence of this problem the equation (1) can also be solved analytically, so that the numerical results can be compared with the exact analytical solution. In order to construct the integral equation, the solution T is written as:

$$T(\bar{r}) = \oint_C \rho(\bar{r}')G(|\bar{r} - \bar{r}'|)\,dC' \qquad (5)$$

where $\rho(\bar{r})$ is an unknown complex source function defined along the boundary C. Imposing the boundary conditions (4) on the proposed solution (5) yields:

$$\oint_C \rho(\bar{r}')G(|\bar{r} - \bar{r}'|)\,dC' = T_0 \quad \bar{r} \in AA' \qquad (6)$$

$$\oint_C \rho(\bar{r}')G(|\bar{r} - \bar{r}'|)\,dC' = 0 \quad \bar{r} \in BB' \qquad (7)$$

$$\frac{\rho(\bar{r})}{2} + \oint \rho(\bar{r}')\nabla_{\bar{r}}\, G(|\bar{r} - \bar{r}'|) \cdot \bar{u}_n\,dC' = 0 \quad r \in AB \text{ and } A'B' \qquad (8)$$

where:

$$\nabla_{\bar{r}} G(|\bar{r}-\bar{r}'|) = \frac{\sqrt{(\omega/D)}}{2}\{ker'_0[|\bar{r}-\bar{r}'|\sqrt{(\omega/D)}]$$
$$+ i\,kei'_0[|\bar{r}-\bar{r}'|\sqrt{(\omega/D)}]\}\frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|}. \quad (9)$$

The relations (6–8) constitute an integral equation for the unknown function $\rho(\bar{r})$. The first term appearing in (8) is caused by the discontinuity of the Green's function for $\bar{r} \to \bar{r}'$. The same problem occurs for potential equations [13]. Once the integral equation has been solved, the function T can be easily calculated by applying the formula (5). It is also possible to calculate the gradient of T by:

$$\nabla T = \oint_C \rho(\bar{r}')\nabla_{\bar{r}} G(|\bar{r}-\bar{r}'|)\mathrm{d}C'. \quad (10)$$

By integrating (10) along a given line, the flux of ∇T is easily found. For a point $\bar{r}$ lying on the boundary C, the relation (10) should be corrected for the normal component $\nabla T \cdot \bar{u}_n$ as has been done for the equation (8).

3. NUMERICAL SOLUTION

In order to start the numerical solution of the integral equation, the boundary C is divided into n intervals ΔC_i. In each interval ΔC_i the function $\rho(\bar{r})$ is replaced by a complex number ρ_i. The integral equation can then be reduced to a linear algebraic set of n unknowns [3], which can be written quite generally as:

$$\sum_{j=1}^{n} a_{ij}\rho_j = b_i. \quad (11)$$

Denoting $\bar{r}_i$ as the centre point of the interval ΔC_i, the coefficients a_{ij} and b_i for $\bar{r}_i \in AA'$ may be written as:

$$a_{ij} = G(|\bar{r}_i - \bar{r}_j|)|\Delta C_i| \quad (12)$$
$$b_i = T_0 \quad (13)$$

where $|\Delta C_i|$ denotes the length of the ith interval. Similar expressions as (12) and (13) can be written if $\bar{r}_i$ is located elsewhere on the boundary. For the diagonal elements a_{ii}, the expression (12) cannot be used because the Green's function (3) tends to infinity if the argument becomes zero. The relation (12) should then be replaced by:

$$a_{ii} = \int_{\Delta C_i} G(|\bar{r}_i - \bar{r}'|)\,\mathrm{d}C' = 2\int_0^{|\Delta C_i|/2} G(x)\mathrm{d}x. \quad (14)$$

By integrating the numerical approximations of the Kelvin functions [14], the integrals (14) are easily evaluated. The algebraic set (11) can now be easily solved by the Gauss elimination method [15].

For the particular geometry shown in Fig. 1, the problem can also be solved analytically. By looking for a y-independent solution, the boundary condition $\nabla T \cdot \bar{u}_n = 0$ is automatically fulfilled. One obtains then:

$$T = A\,\mathrm{e}^{\sqrt{(j\omega/D)}x} + B\,\mathrm{e}^{-\sqrt{(j\omega/D)}x} \quad (15)$$

where:

$$A = \frac{T_0}{1-\exp[2\sqrt{(j\omega/D)}a]},$$
$$B = -A\exp[2\sqrt{(j\omega/D)}a]. \quad (16)$$

In order to check the accuracy of the numerical method, the flux of T through AA' was calculated numerically. These results are compared with the analytical expression:

$$b\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)_{x=0} = \sqrt{(j\omega/D)}b\,\frac{1+\exp[2\sqrt{(j\omega/D)}a]}{1-\exp[2\sqrt{(j\omega/D)}a]}. \quad (17)$$

Figure 2 represents the real and imaginary part of (17) for $b = a = 10$ and $D = 1$. Some numerical results obtained for $n = 20$ and $n = 40$ are also shown. One sees that the accuracy is fairly good even for a relatively low value of n.

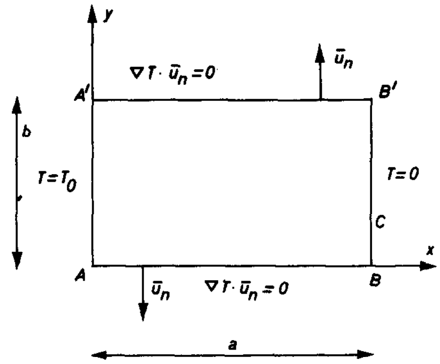

FIG. 1. Rectangular geometry used to establish the integral equation for the diffusion equation under AC conditions.

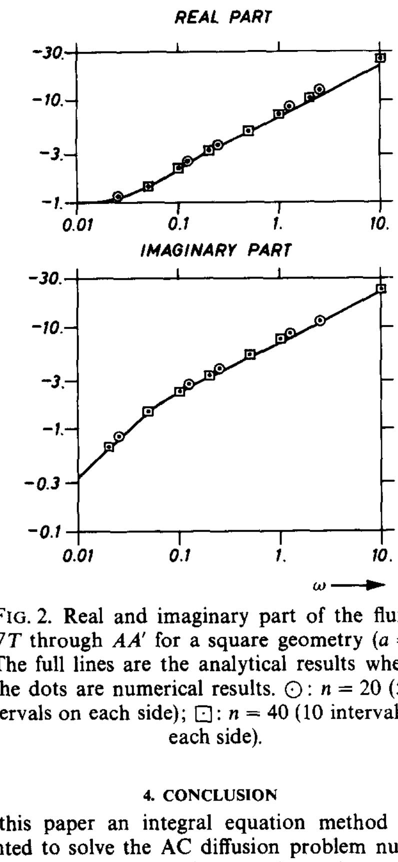

FIG. 2. Real and imaginary part of the flux of ∇T through AA' for a square geometry ($a = b$). The full lines are the analytical results whereas the dots are numerical results. ⊙: $n = 20$ (5 intervals on each side); ⊡: $n = 40$ (10 intervals on each side).

4. CONCLUSION

In this paper an integral equation method has been presented to solve the AC diffusion problem numerically. The two-dimensional diffusion equation is then reduced to a one-dimensional complex integral equation which largely simplifies the numerical solution. The method has been checked for a particular geometry by comparing with the exact analytical results. It was found that a good accuracy could be obtained even for moderate values of the number n of intervals and for high values of the frequency ω.

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